From cellular automaton to difference equation: a general transformation method which preserves time evolution patterns

2001 J. Phys. A: Math. Gen. 34 L371

(http://iopscience.iop.org/0305-4470/34/25/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **34** (2001) L371–L379 PII: S0305-4470(01)18243-0

LETTER TO THE EDITOR

From cellular automaton to difference equation: a general transformation method which preserves time evolution patterns

A Nobe, J Satsuma and T Tokihiro

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

Received 20 October 2000, in final form 17 May 2001
Published 15 June 2001
Online at stacks.iop.org/JPhysA/34/L371

Abstract
We propose a general method to construct a partial difference equation which preserves any time evolution patterns of a cellular automaton. The method is based on inverse ultradiscretization with filter functions.

PACS numbers: 05.45.Y, 02.10.E, 87.17.-d

1. Introduction

Ultradiscretization is a procedure by which we can transform a discrete equation to a cellular automaton (CA) [1, 2]. In the procedure, we take a limit of a parameter which exists in the equation and obtain an equation which is closed under discrete values. Since a solution of the original equation also depends on the parameter, by taking this limit for the solution, we simultaneously obtain the analytical expression for the time evolution patterns of the CA. The patterns thus obtained naturally preserve the features of the original solutions. A typical example is the soliton CA proposed by Takahashi and one of the authors (JS) [3].

On the other hand, when we try its inverse process—inverse ultradiscretization—we encounter a serious difficulty [4]. The inverse ultradiscretization is usually performed with the following steps.

- We rewrite the time evolution rule of a CA to a piecewise linear equation with max and plus algebra.
- By introducing a parameter ε, we replace $\max[a, b]$ by $\varepsilon \log[\mathrm{e}^{a/\varepsilon} + \mathrm{e}^{b/\varepsilon}]$.

For example, in the rule 90 elementary cellular automaton (ECA), the value of the jth site at time step $t + 1$ (u_j^{t+1}) is determined by the values at the previous time step t as [5]

$$u_j^{t+1} \equiv u_{j-1}^t + u_{j+1}^t \qquad \text{modulo } 2. \tag{1}$$

One of the simplest piecewise linear equations corresponding to equation (1) is

$$u_j^{t+1} = \max[u_{j-1}^t - u_{j+1}^t, u_{j+1}^t - u_{j-1}^t]. \tag{2}$$

Hence the corresponding (inverse ultradiscretized) difference equation is obtained as

$$u_j^{t+1} = \varepsilon \log[\exp[(u_{j-1}^t - u_{j+1}^t)/\varepsilon] + \exp[(u_{j+1}^t - u_{j-1}^t)/\varepsilon]] - \varepsilon \log 2 \tag{3}$$

or, replacing $\exp(u_j^t/\varepsilon)$ with U_j^t,

$$U_j^{t+1} = \frac{1}{2}\left(\frac{U_{j-1}^t}{U_{j+1}^t} + \frac{U_{j+1}^t}{U_{j-1}^t}\right). \tag{4}$$

Here the constant $-\varepsilon \log 2$, vanishing in the ultradiscrete limit ($\varepsilon \to +0$), is introduced so that $u_j^t \equiv 0$ ($\forall\, j, t$) remains a solution in the inverse ultradiscretized process. The rule 90 ECA has a time evolution pattern which shows a typical fractal structure [5]. However, equation (3) does not have a solution which preserves the fractal structure as long as the parameter ε is finite [4]. This suggests that equation (2) is not an appropriate piecewise linear equation for the rule 90 ECA.

In this Letter, we give a universal method of inverse ultradiscretization which preserves *any* time evolution pattern of *any* CA. In section 2, we define a notion of a *stable piecewise linear equation* for a CA. In section 3, we present a general method to construct a stable piecewise linear equation from a given CA. In section 4, we generalize the results obtained in the previous section with the *filtration* function and examine the inverse ultradiscretization of the equation which preserves the original time evolution pattern of the CA as long as the parameter ε is small enough. Section 5 is devoted to concluding remarks.

2. Stable piecewise linear equation for a CA

Let us consider a CA, each site of which takes $L + 1$ distinct values ($L \in \mathbb{Z}$). Without loss of generality, we may assume that the set of these values is $\mathcal{S}_L := \{0,\ 1, \ldots, L\}$. We also assume that the CA is 1(space) + 1(time) dimensional for simplicity. Generalization to higher-dimensional cases is straightforward. We denote the time evolution rule of the CA by

$$u_j^{t+1} = F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t) \tag{5}$$

where $u_j^t \in \mathcal{S}_L$ is the value of $j \in \mathbb{Z}$ site at time step $t \in \mathbb{Z}$, and F is a map from $\underbrace{\mathcal{S}_L \times \mathcal{S}_L \times \cdots \mathcal{S}_L}_{M\ \text{times}}$ to $\mathcal{S}_L$. We consider an equation associated with equation (5):

$$x_j^{t+1} = K_F(x_{j-k+1}^t, x_{j-k+2}^t, \ldots, x_{j-k+M}^t) \tag{6}$$

where $x_j^t \in \mathbb{R}$ is the dependent variable of the equation with independent space ($j \in \mathbb{Z}$) and time ($t \in \mathbb{Z}$) variables, and K_F denotes a piecewise linear map[1] from $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}}_{M\ \text{times}}$ to $\mathbb{R}$.

Definition 1. *A piecewise linear map* $K_F : \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}}_{M\ \text{times}} \to \mathbb{R}$ *is called a piecewise linear map associated with the CA given by (5) when it coincides with* F *on* $\mathcal{S}_L$*, that is,*

$$K_F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t) = F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t) \qquad \forall\, u_j^t \in \mathcal{S}_L.$$

For a given CA, there are in principle an infinite number of piecewise linear maps associated with it. However, in order to obtain a difference equation which shows a similar behaviour to the CA, the piecewise linear map should satisfy a certain property.

[1] By ‘a piecewise linear map’, we denote that the domain of the map is a Euclidean simplicial complex and the map is linear on each simplex.

Definition 2. *The piecewise linear map K_F associated with the CA given by (5) is called stable if there exists a positive number δ ($0 < \delta < \frac{1}{2}$) such that if, $\forall\, j$, $|u_j^t - x_j^t| < \delta$ then*

$$|K_F(x_{j-k+1}^t, x_{j-k+2}^t, \ldots, x_{j-k+M}^t) - F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t)| < \delta.$$

From the definition 2, we find that equation (6) preserves *any* time evolution pattern of the CA in the sense that, when the initial values of equation (6) approximately take values in $\mathcal{S}_L$ with a tolerance of δ, the time evolution pattern of equation (6) coincides with that of (5) if we round off the dependent variables in the pattern. Therefore we claim that, *in the inverse ultradiscretization, the piecewise linear map obtained in the first step should be stable with respect to the CA*. For example, the right-hand side of equation (2) is a piecewise linear map associated with the CA (1). However, the map is not stable because it does not satisfy the condition in the definition 2 at $(u_{j-1}^t, u_{j+1}^t) = (1, 0)$. In the next section, we shall show a method to construct a stable piecewise linear map associated with a given CA.

3. Stable piecewise linear map associated with a CA

We define real functions $\chi(x : a_0, a_1)$ and $X(x : a_0, a_1, a_2)$ as

$$\chi(x : a_0, a_1) := \frac{\max[x, a_1] - \max[x, a_0]}{a_1 - a_0} \tag{7}$$

$$X(x : a_0, a_1, a_2) := -\chi(x : a_0, a_1) + \chi(x : a_1, a_2). \tag{8}$$

Here we assume that $a_j \in \mathbb{R}$ ($j = 0, 1, 2$) and satisfy the inequalities $a_j < a_{j+1}$ ($j = 0, 1$). The function $X(x : a_0, a_1, a_2)$ is written explicitly as

$$X(x : a_0, a_1, a_2) = \begin{cases} 0 & \text{for} \quad x \leqslant a_0 \\ \dfrac{x - a_0}{a_1 - a_0} & \text{for} \quad a_0 < x \leqslant a_1 \\ \dfrac{a_2 - x}{a_2 - a_1} & \text{for} \quad a_1 < x \leqslant a_2 \\ 0 & \text{for} \quad a_2 < x. \end{cases} \tag{9}$$

The function $X(x : a_0, a_1, a_2)$ is continuous and piecewise linear with respect to x.

For given data $\{A_{i_1,i_2,\ldots,i_n} \in \mathbb{R}\}$ ($i_j = 0, 1, 2, \ldots, L_j \quad j = 1, 2, \ldots, n$) and $\{a_i^{(j)} \in \mathbb{R}\}$ ($i = -1, 0, 1, \ldots, L_j + 1 \quad j = 1, 2, \ldots, n$), we define a continuous and piecewise linear function with n independent variables $x_1, x_2, \ldots, x_n$ as

$$\Theta(x_1, x_2, \ldots, x_n : \{a_i^{(j)}\} : \{A_{i_1,i_2,\ldots,i_n}\}) \\ := \sum_{i_1=0}^{L_1} \sum_{i_2=0}^{L_2} \cdots \sum_{i_n=0}^{L_n} A_{i_1,i_2,\ldots,i_n} \min_{1 \leqslant j \leqslant n} [X(x_j : a_{i_j-1}^{(j)}, a_{i_j}^{(j)}, a_{i_j+1}^{(j)})] \tag{10}$$

where we assume that $a_{-1}^{(j)} = -\infty$, $a_{L_j+1}^{(j)} = +\infty \ \forall\, j$, and $a_i^{(j)} < a_{i+1}^{(j)} \ \forall\, i, \forall\, j$. An example of the function $\Theta(x_1, x_2, \ldots, x_n : \{a_i^{(j)}\} : \{A_{i_1,i_2,\ldots,i_n}\})$ is illustrated in figure 1.

The following proposition is proved easily from (9):

Proposition 1.

$$\Theta(a_{k_1}^{(1)}, a_{k_2}^{(2)}, \ldots, a_{k_n}^{(n)} : \{a_i^{(j)}\} : \{A_{i_1,i_2,\ldots,i_n}\}) = A_{k_1,k_2,\ldots,k_n} \ (k_j = 0, 1, 2, \ldots, L_j) \qquad \forall\, j. \tag{11}$$

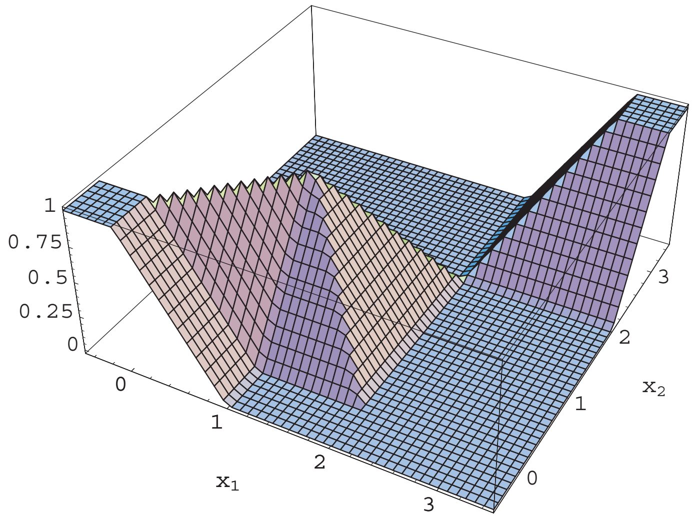

Figure 1. The function (10) with two independent variables x_1, x_2 for $a_0^{(1)} = a_0^{(2)} = 0$, $a_1^{(1)} = a_1^{(2)} = 1, a_2^{(1)} = a_2^{(2)} = 2, a_3^{(1)} = a_3^{(2)} = 3, L_1 = L_2 = 3$ and $A_{i_1,i_2} = 0\,(i_1, i_2 = 0, 1, 2, 3)$ except for $A_{0,0} = A_{1,1} = A_{3,3} = 1$.

(This figure is in colour only in the electronic version)

Using the function (10), we can construct a stable piecewise linear function associated with a given CA. First we give a simple example to show the idea of obtaining a stable piecewise linear function. Let us consider a map M from {0, 1, 2, 3, 4} to {0, 1, 2, 3, 4}:

$$M : 0 \to 0 \qquad 1 \to 2 \qquad 2 \to 4 \qquad 3 \to 2 \qquad 4 \to 0. \tag{12}$$

A natural piecewise linear map associated with M would be $F_M^{(0)}$, which is defined as

$$\begin{aligned} x_{t+1} &= F_M^{(0)}(x_t) \\ &:= \begin{cases} 2x_t & \text{for} \quad 0 \leqslant x_t \leqslant 2 \\ 2 - 2x_t & \text{for} \quad 2 < x_t \leqslant 4. \end{cases} \end{aligned} \tag{13}$$

However the map $F_M^{(0)}$ is not *stable* in the sense of the definition 1. To obtain a stable piecewise linear map F_M, it is sufficient to impose the following condition on F_M:

- F_M is a continuous piecewise linear equation and

$$F_M(x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant x \leqslant \Delta \\ 2 & \text{for} \quad 1 - \Delta \leqslant x \leqslant 1 + \Delta \\ 4 & \text{for} \quad 2 - \Delta \leqslant x \leqslant 2 + \Delta \\ 2 & \text{for} \quad 3 - \Delta \leqslant x \leqslant 3 + \Delta \\ 0 & \text{for} \quad 4 - \Delta \leqslant x \leqslant 4. \end{cases} \tag{14}$$

Here $0 < \Delta < 1/2$. It is obvious that the above condition uniquely determines the map which is a piecewise linear stable map associated with M. We generalize the above construction to CAs. As in section 2, we consider a (1 + 1)-dimensional CA which takes values on S_L. The time evolution rule is expressed as

$$u_j^{t+1} = F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t) \tag{15}$$

where u_j^t is the dependent variable on jth site at time step t. We define $2L$ points in the interval $I := [0, L]$ with a positive parameter Δ ($0 < \Delta < \frac{1}{2}$) as

$$p_j = \left[\frac{j+1}{2}\right] + (-1)^j \Delta \tag{16}$$

where $[\cdot]$ denotes the largest integer which does not exceed '$\cdot$'. Hence, we have

$$\begin{aligned} &p_0 = \Delta \qquad p_1 = 1 - \Delta \qquad p_2 = 1 + \Delta \qquad p_3 = 2 - \Delta \\ &\ldots \qquad p_{2L-2} = L - 1 + \Delta \qquad p_{2L-1} = L - \Delta. \end{aligned}$$

We also define that $p_{-1} := -\infty$ and $p_{2L} := +\infty$. With these p_j, we further define a piecewise linear function with M independent variables K_F: $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{M \text{ times}} \to \mathbb{R}$, as

$$K_F(x_1, x_2, \ldots, x_M) = \Theta(x_1, x_2, \ldots, x_M : \{a_i^{(j)}\} : \{A_{i_1,\ldots,i_M}\}) \tag{17}$$

where $a_i^{(j)} = p_i,\ L_j = 2L - 1\ \ (i = -1, 0, 1, \ldots, 2L,\ \ j = 1, 2, \ldots, M)$ and

$$\begin{aligned} 0 < p_0 < \tfrac{1}{2} < p_1 < 1 < p_2 < \tfrac{3}{2} < p_3 < 2 < p_4 < \tfrac{5}{2} < p_5 < \cdots \\ < p_{2L-2} < L - \tfrac{1}{2} < p_{2L-1} < L. \end{aligned}$$

Denoting by $[[\cdot]]$ the integer to which '$\cdot$' is rounded, and noticing that $[[p_{2m-1}]] = [[p_{2m}]] = m$, we set

$$\begin{aligned} A_{i_1,i_2,\ldots,i_M} &= F([[a_{i_1}^{(1)}]], [[a_{i_2}^{(2)}]], \ldots, [[a_{i_M}^{(M)}]]) \\ &= F([[p_{i_1}]], [[p_{i_2}]], \ldots, [[p_{i_M}]]). \end{aligned}$$

Then, from proposition 1, we find the following. In the difference equation

$$u_j^{t+1} = K_F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t) \tag{18}$$

the function K_F is a continuous and piecewise linear function on $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{M\text{times}}$. It takes the same values not only on the lattice $\underbrace{\mathcal{S}_L \times \mathcal{S}_L \times \cdots \times \mathcal{S}_L}_{M\text{times}}$ but also in the Δ neighbourhood as those of $F(u_{j-k+1}^t, u_{j-k+2}^t, \ldots, u_{j-k+M}^t)$. Hence we have:

Theorem 1. *The function K_F (17) gives a stable piecewise linear map associated with the CA (15).*

Since the time evolution rule F is arbitrary, we can construct a stable piecewise linear map for any CA. Extension to higher-dimensional CAs is straightforward. Here we give a few examples.

For the ECAs, we consider four points on $\mathbb{R}$

$$p_{-1} = -\infty \qquad p_0 = \Delta \qquad p_1 = 1 - \Delta \qquad p_2 = +\infty.$$

We define

$$\begin{aligned} X_0(x) &:= X(x : p_{-1}, p_0, p_1) \\ &= X(x : -\infty, \Delta, 1 - \Delta) \\ &= -\chi(x : -\infty, \Delta) + \chi(x : \Delta, 1 - \Delta) \\ &= -0 + \frac{\max[x, 1 - \Delta] - \max[x, \Delta]}{(1 - \Delta) - (\Delta)}. \end{aligned}$$

The function $X_0(x)$ is continuous, piecewise linear and takes its maximum value unity for $x \leqslant \Delta$ and becomes zero for $x \geqslant 1-\Delta$. Similarly, we define

$$\begin{aligned} X_1(x) &:= X(x : p_0, p_1, p_2) \\ &= X(x : \Delta, 1-\Delta, +\infty) \\ &= -\chi(x : \Delta, 1-\Delta) + \chi(x : 1-\Delta, +\infty) \\ &= -\frac{\max[x, 1-\Delta] - \max[x, \Delta]}{(1-\Delta)-(\Delta)} + 1. \end{aligned}$$

The function $X_1(x)$ is also continuous, piecewise linear and takes zero for $x \leqslant \Delta$ and unity for $x \geqslant 1-\Delta$.

With these functions, K_F is written as

$$K_F(x, y, z) = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} A_{i,j,k} \min[X_i(x), X_j(y), X_k(z)]. \tag{19}$$

If the time evolution rule of an ECA is given by

$$u_n^{t+1} = F_{\mathrm{ECA}}(u_{n-1}^t, u_n^t, u_{n+1}^t)$$

then the coefficients $A_{i,j,k}$ are expressed as $A_{i,j,k} = F_{\mathrm{ECA}}(i, j, k)$, and we find that the piecewise linear difference equation for the ECA is given by

$$\begin{aligned} u_n^{t+1} &= K_F(u_{n-1}^t, u_n^t, u_{n+1}^t) \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} F_{\mathrm{ECA}}(i, j, k) \min[X_i(u_{n-1}^t), X_j(u_n^t), X_k(u_{n+1}^t)]. \end{aligned} \tag{20}$$

For the game of life (a typical two-dimensional CA) [6], we have the piecewise linear difference equation

$$\begin{aligned} u_{n,m}^{t+1} = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\cdots\sum_{i_9=0}^{1} & F_{\mathrm{LG}}(i_1, i_2, i_3, \ldots, i_9) \\ & \times \min[X_{i_1}(u_{n-1,m-1}^t), X_{i_2}(u_{n-1,m}^t), X_{i_3}(u_{n-1,m+1}^t), \ldots, X_{i_9}(u_{n+1,m+1}^t)] \end{aligned} \tag{21}$$

where $u_{n,m}^{t+1} = F_{\mathrm{LG}}(u_{n-1,m-1}^t, u_{n-1,m}^t, u_{n-1,m+1}^t, \ldots, u_{n+1,m+1}^t)$ denotes the time evolution rule of the game of life. More detailed results of (21) are discussed in [7].

As the final example of this section, we consider a CA which takes three distinct values, zero, unity and two. We assume that the time evolution rule of the CA is given by

$$u_n^{t+1} = F_{\mathrm{tri}}(u_{n-1}^t, u_n^t, u_{n+1}^t).$$

From the general arguments stated above, we set

$$\begin{aligned} &p_{-1} = -\infty \qquad p_0 = \Delta \qquad p_1 = 1-\Delta \\ &p_2 = 1+\Delta \qquad p_3 = 2-\Delta \qquad p_4 = +\infty. \end{aligned}$$

The following four piecewise linear functions are necessary for constructing the piecewise linear map.

$$\begin{aligned} X_0(x) &:= X(x : p_{-1}, p_0, p_1) \\ &= X(x : -\infty, \Delta, 1-\Delta) \\ &= -\chi(x : -\infty, \Delta) + \chi(x : \Delta, 1-\Delta) \\ &= -0 + \frac{\max[x, 1-\Delta] - \max[x, \Delta]}{(1-\Delta)-(\Delta)}. \end{aligned}$$

$$\begin{aligned} X_1(x) &:= X(x : p_0, p_1, p_2) \\ &= X(x : \Delta, 1-\Delta, 1+\Delta) \\ &= -\chi(x : \Delta, 1-\Delta) + \chi(x : 1-\Delta, 1+\Delta) \\ &= -\frac{\max[x, 1-\Delta] - \max[x, \Delta]}{(1-\Delta)-(\Delta)} + \frac{\max[x, 1+\Delta] - \max[x, 1-\Delta]}{(1+\Delta)-(1-\Delta)}. \end{aligned}$$

This function vanishes for $x \leqslant \Delta$ and $1 + \Delta \leqslant x$, and takes its maximum value unity at $x = 1 - \Delta$.

$$\begin{aligned} X_2(x) &:= X(x : p_1, p_2, p_3) \\ &= X(x : 1-\Delta, 1+\Delta, 2-\Delta) \\ &= -\chi(x : 1-\Delta, 1+\Delta) + \chi(x : 2-\Delta, 1+\Delta) \\ &= -\frac{\max[x, 1+\Delta] - \max[x, 1-\Delta]}{(1+\Delta)-(1-\Delta)} + \frac{\max[x, 2-\Delta] - \max[x, 1+\Delta]}{(2-\Delta)-(1+\Delta)}. \end{aligned}$$

This function vanishes for $x \leqslant 1 - \Delta$ and $2 - \Delta \leqslant x$, and takes its maximum value unity at $x = 1 + \Delta$.

$$\begin{aligned} X_3(x) &:= X(x : p_2, p_3, p_4) \\ &= X(x : 1+\Delta, 2-\Delta, +\infty) \\ &= -\chi(x : 1+\Delta, 2-\Delta) + \chi(x : 2-\Delta, +\infty) \\ &= -\frac{\max[x, 2-\Delta] - \max[x, 1+\Delta]}{(2-\Delta)-(1+\Delta)} + 1. \end{aligned}$$

The difference function is given by

$$\begin{aligned} u_n^{t+1} &= K_F(u_{n-1}^t, u_n^t, u_{n+1}^t) \\ &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} F_{\mathrm{tri}}([[p_i]], [[p_j]], [[p_k]]) \min[X_i(x), X_j(y), X_k(z)]. \end{aligned}$$

Note that $[[p_0]] = 0$, $[[p_1]] = 1$, $[[p_2]] = 1$, $[[p_3]] = 2$.

4. Generalization and inverse ultradiscretization

In the previous section, we have proposed a method to construct a stable piecewise linear difference equation associated with a CA. This method can be applied to any kind of CA. However, it is not a unique method to obtain a stable piecewise linear equation. In this section, we introduce a filter function $\Xi(x)$ and generalize the above method. Let us consider $(L+1)$-valued CAs as in the previous sections. We also assume that its time evolution is given by (5). The function $\Xi(x)$ is continuous, piecewise linear on $\mathbb{R}$ and defined by

$$\Xi(x) := \sum_{j=0}^{L} X_j(x) \tag{22}$$

where

$$X_j(x) := \begin{cases} 0 & \text{for} \quad x \leqslant j + \Delta_j^- \\ \dfrac{x - j - \Delta_j^-}{\Delta_j^+ - \Delta_j^-} & \text{for} \quad j + \Delta_j^- < x \leqslant j + \Delta_j^+ \\ 1 & \text{for} \quad j + \Delta_j^+ < x \end{cases} \tag{23}$$

or, equivalently,

$$X_j(x) = \frac{\max[0, x - j - \Delta_j^-] - \max[0, x - j - \Delta_j^+]}{\Delta_j^+ - \Delta_j^-}. \tag{24}$$

Here Δ_j^-, Δ_j^+ $(j = 0, 1, \ldots, L)$ are positive parameters which satisfy

$$0 < \Delta_j^- < \tfrac{1}{2} < \Delta_j^+ < 1 \qquad (j = 0, 1, \ldots, L).$$

Next we consider a piecewise linear map G_F associated with the CA. The map G_F is not necessarily stable. An example is given by Θ (10). We set $a_i^{(j)} = i \ \forall j \ (i = 0, 1, \ldots, L)$ and $A_{i_1,i_2,\ldots,i_n} = F(i_1, i_2, \ldots, i_n)$. Then we have

$$\begin{aligned} G_F(x_1, x_2, \ldots, x_n) &= \Theta(x_1, x_2, \ldots, x_n : \{a_i^{(j)}\} : \{A_{i_1,i_2,\ldots,i_n}\}) \\ &= \sum_{i_1=0}^{L}\sum_{i_2=0}^{L}\cdots\sum_{i_n=0}^{L} F(i_1, i_2, \ldots, i_n) \min_{1\leqslant j\leqslant n} [X(x_j : a_{i_j-1}^{(j)}, a_{i_j}^{(j)}, a_{i_j+1}^{(j)})]. \end{aligned} \tag{25}$$

The stable piecewise linear function is obtained by composition of G_F and Ξ, namely,

$$\begin{aligned} K_F(x_1, x_2, \ldots, x_n) &= (G_F \circ \Xi)(x_1, x_2, \ldots, x_n) \\ &= G_F(\Xi(x_1), \Xi(x_2), \ldots, \Xi(x_n)) \end{aligned} \tag{26}$$

or

$$\begin{aligned} K'_F(x_1, x_2, \ldots, x_n) &= (\Xi \circ G_F)(x_1, x_2, \ldots, x_n) \\ &= \Xi(G_F(x_1, x_2, \ldots, x_n)). \end{aligned} \tag{27}$$

Clearly, the function considered in the previous section is a special case of (26). Note that $\Xi \circ G_F$ and $G_F \circ \Xi$ are essentially the same as dynamical maps. (Expression (26) is slightly more general, since we can introduce a different kind of filter function Ξ for each x_j.)

Finally we briefly discuss inverse ultradiscretization. Rigorous examination for some concrete CAs will be performed in a forthcoming paper [8]. Since $\min[a, b] = -\max[-a, -b]$, we have only to replace max with some appropriate continuous function. For soliton cellular automata, the relation

$$\max[a, b] = \lim_{\varepsilon\to+0} \varepsilon \log(\mathrm{e}^{a/\varepsilon} + \mathrm{e}^{b/\varepsilon}) \tag{28}$$

is found to be useful because the soliton solutions are expressed with linear combinations of exponential functions. However, there are many limiting procedures to yield $\max[a, b]$, such as

$$\max[a, b] = \lim_{\varepsilon\to+0} a + \frac{b - a}{1 + \exp[(a - b)/\varepsilon]} \tag{29}$$

and, for $a, b > 0$,

$$\max[a, b] = \lim_{N\to+\infty} \left(\frac{a^N + b^N}{2}\right)^{\frac{1}{N}}. \tag{30}$$

(It may be interesting to note that the function $f(N : a, b) := (\frac{a^N+b^N}{2})^{\frac{1}{N}}$ is an increasing function of N and $f(-\infty : a, b) = \min[a, b]$, $f(0 : a, b) = \sqrt{ab}$.) We do not know which is the best replacement. It depends on the case. However, if the parameter is close enough to the limiting value, the difference equation obtained from the stable piecewise linear equation shows a similar pattern to that of the original CA.

5. Conclusion

In this paper, we have defined a stable piecewise linear map associated with a given CA and discussed its importance in inverse ultradiscretization. We have presented a general method to construct a stable piecewise linear map from a CA. Inverse ultradiscretization of the map shows a similar time evolution pattern to that of the CA. These patterns are stable against fluctuations as long as the parameter ε is small enough. We have also given a general formalism to construct a stable map with the filter functions. Concrete examples for the elementary CA and the game of life will be reported elsewhere with detailed analysis [7, 8].

The aim of our research is to answer Wolfram's ninth problem, 'what is the correspondence between cellular automata and continuous systems?' [9]. Several contributions to this problem have been reported. A system of ten coupled nonlinear partial differential equations has been proposed to simulate two-dimensional nine-neighbour square lattice CAs [10]. Since the C^∞ bump functions are used in this approach, the resulting partial differential equations look fairly artificial, though this approach can be applicable to an arbitrary CA. For so-called integrable cellular automata, ultradiscretization successfully gives a direct link between CAs and integrable partial differential equations. However, for nonintegrable systems, we have not established a direct link with ultradiscretization. For reaction–diffusion equations, the lattice-gas cellular automata can lead to discrete lattice Boltzmann equations and, by coarse graining and a scaling procedure, can produce partial differential equations [11]. In general, however, dynamics of the resulting equations can be somewhat different from the dynamics of the corresponding lattice-gas cellular automata. It would be significant if we could establish a unified approach to establish relations between CAs and partial differential equations. We hope that the notions and the methods presented here will make some contribution to solving the ninth problem.

The authors are grateful to Dr Tomonori Watanabe for helpful discussions. This paper is partially supported by a Grant-in-Aid from the Japan Ministry of Education, Science and Culture.

References

[1] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247
[2] Matsukidaira J, Satsuma J, Takahashi D, Tokihiro T and Torii M 1997 *Phys. Lett.* A **255** 287
[3] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 3514
[4] Nobe A 1999 *Master Thesis* University of Tokyo
[5] Wolfram S (ed) 1986 *Theory and Applications of Cellular Automata* (Singapore: World Scientific)
[6] See for example, Gardner M 1983 *Wheels, Life and Other Mathematical Amusements* (San Francisco: Freeman)
[7] Watanabe T, Tokihiro T and Satsuma J From the game of life to difference equations *Preprint*
[8] Nobe A, Satsuma J and Tokihiro T Stable difference equations associated with elementary cellular automata *Preprint*
[9] Wolfram S 1985 *Phys. Scr.* T **9** 170–83
(This paper has been reprinted in Wolfram S 1994 *Cellular Automata and Complexity* (Reading, MA: Addison-Wesley))
[10] Omohundro S 1984 *Physica* D **10** 128
[11] Boon J P, Dab D, Kapral R and Lawniczak A 1996 *Phys. Rep.* **273** 55